The Kähler cone of a compact hyperkähler manifold

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Abstract

This paper is a sequel to [11]. We study a number of questions only touched upon in [11] in more detail. In particular: What is the relation between two birational compact hyperkähler manifolds? What is the shape of the cone of all Kähler classes on such a manifold? How can the birational Kähler cone be described? Most of the results are motivated by either the well-established two-dimensional theory, i.e. the theory of K3 surfaces, or the theory of Calabi-Yau threefolds and string theory.

1 Introduction

Let X be a K3 surface and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a class in the positive cone with $\alpha.C \neq 0$ for any smooth rational curve C. Then there exist smooth rational curves C_i such that for the effective cycle $\Gamma := \Delta + \sum C_i \times C_i \subset X \times X$ (where Δ is the diagonal) the class $[\Gamma]_*(\alpha)$ is a Kähler class. This is a consequence of the transitivity of the action of the Weyl-group on the set of chambers in the positive cone and the description of the Kähler cone of a K3 surface due to Todorov and Siu (cf. [1]). In higher dimension the following result was proved in [11, Cor. 5.2]:

— *Let X be a compact hyperkähler manifold of dimension $2n$ and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a very general class in the positive cone. Then there exists another compact hyperkähler manifold X' and an effective cycle $\Gamma := Z + \sum Y_i \subset X \times X'$ of pure dimension $2n$ such that $X \leftarrow Z \rightarrow X'$ defines a birational correspondence, the projections $Y_i \rightarrow X$, $Y_i \rightarrow X'$ are of positive dimensions, and $[\Gamma]_*(\alpha)$ is a Kähler class on X'.*

In this article we present a few applications of this result. In Sect. 2 we will show the following theorem:

— *Let X and X' be compact hyperkähler manifolds. If X and X' are birational then there exist two smooth proper morphisms $\mathcal{X} \rightarrow S$ and $\mathcal{X}' \rightarrow S$ with S smooth and one-dimensional such that for a distinguished point $0 \in S$ one has $\mathcal{X}_0 \cong X$ and $\mathcal{X}'_0 \cong X'$ and such that there exists an isomorphism $\mathcal{X}|_{S \setminus \{0\}} \cong \mathcal{X}'|_{S \setminus \{0\}}$.*

In particular, X and X' are deformation equivalent and, hence, diffeomorphic. This immediately proves that the Betti numbers of X and X' coincide and, more precisely, their Hodge

structures are isomorphic. The result was proved in [11] for projective hyperkähler manifolds. The proof given here is different even in the projective case. The fact about the Betti numbers also follows from a theorem of Batyrev [2] and Kontsevich [7]. They show that if X and X' are birational smooth projective varieties with trivial canonical bundle, then $b_i(X) = b_i(X')$ (cf. [2]) and their Hodge structures are isomorphic (cf. [7]). For hyperkähler manifolds the above result proves that the cohomology rings of birational hyperkähler manifolds are isomorphic. This is no longer true for arbitrary Calabi-Yau manifolds.

In Sect. 3 we give a description of the Kähler cone of a compact hyperkähler manifold very much in the spirit of the known one for K3 surfaces (Prop. 3.2):

— *Let X be a compact hyperkähler manifold. A class $\alpha \in H^{1,1}(X, \mathbb{R})$ is in the closure $\overline{\mathcal{K}}_X$ of the cone of all Kähler classes if and only if α is in the closure $\overline{\mathcal{C}}_X$ of the positive cone and $\alpha.C \geq 0$ for any rational curve $C \subset X$.*

The arguments applied to the K3 surface situation differ from the known ones in as much as they make no use of the Global Torelli Theorem.

Any irreducible curve of negative self-intersection on a K3 surface is a smooth rational curve. In higher dimensions one can prove the following generalization (Prop. 5.3):

— *Let C be an irreducible curve in a compact hyperkähler manifold X. If $q_X([C]) < 0$, then C is contained in a uniruled subvariety $Y \subset X$.*

Here, q_X is the quadratic form on $H^{1,1}(X) \cong H^{2n-1,2n-1}(X)$ introduced by Beauville and Bogomolov. Stronger versions of the result are conjectured in Sect. 5.

Due to the existence of birational maps between compact hyperkähler manifolds which do not extend to isomorphisms, one has in higher dimensions the notion of the birational Kähler cone. The birational Kähler cone of Calabi-Yau threefolds has been intensively studied, partly motivated by mirror symmetry, in [12, 14]. Analogously to the description of the Kähler cone, but replacing curves by divisors, one has (Prop. 4.2):

— *The closure of the birational Kähler cone is the set of all classes $\alpha \in \overline{\mathcal{C}}_X$ such that $q_X(\alpha, [D]) \geq 0$ for all uniruled divisors D.*

The present article is the third version of a paper that had originally been prepared for publication in the proceedings of the conference on the occasion of F. Hirzebruch's 70th birthday in Warsaw 1998. As all results of this paper depend on the projectivity criterion Thm. 3.11 in [11] whose proof in [11] is not correct, it has never been published. In the meantime the projectivity criterion for hyperkähler manifolds could be proved by using a new theorem of Demailly and Paun (see the Erratum in [10]). So, all results of this paper are fully proved now. The second version of the article contained a number of basic facts on

the Beauville-Bogomolov quadratic form. They are of independent interest, but not directly relevant to the main results of the article. So we have decided to omit them in this version.

Notations. In this paper a compact hyperkähler manifold is a simply connected Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate holomorphic two-form σ. Equivalently, it is a compact complex manifold that admits a Kähler metric with holonomy $\mathrm{Sp}(n)$, where $\dim(X) = 2n$. If X is a compact hyperkähler manifold then there exists a natural quadratic form q_X of index $(3, b_2(X) - 3)$ on $H^2(X, \mathbb{Z})$ with the property that $q_X(\alpha)^n = \int_X \alpha^{2n}$ up to a scalar factor (cf. [3],[4]). The *positive cone* $\mathcal{C}_X$ is the connected component of the cone $\{\alpha \in H^{1,1}(X, \mathbb{R}) | q_X(\alpha) > 0\}$ that contains the *Kähler cone* $\mathcal{K}_X$ of all Kähler classes.
The word "birational" is used even when the manifolds in question are not projective; what is meant in this case, of course, is "bimeromorphic". I hope this will cause no confusion.

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2 Birational hyperkähler manifolds

Let $f : X' - - \rightarrow X$ be a birational map of compact complex manifolds that induces an isomorphism on open sets whose complements are analytic subsets of codimension ≥ 2. Then $f^* : H^2(X, \mathbb{R}) \cong H^2(X', \mathbb{R})$ and $f^* : \mathrm{Pic}(X) \cong \mathrm{Pic}(X')$, where $(f^*)^{-1} = (f^{-1})^*$.

If X and X' are projective and the pull-back f^*L of an ample line bundle L on X is an ample line bundle on X', then, obviously, f extends to an isomorphism $X \cong X'$. The analytic analogue was proved by Fujiki [8]: If X and X' are Kähler and the pull-back $f^*(\alpha) \in H^2(X', \mathbb{R})$ of a Kähler class $\alpha \in H^2(X, \mathbb{R})$ is a Kähler class on X', then f extends to an isomorphism $X' \cong X$. A slight modification of Fujiki's arguments also proves:

Proposition 2.1 — *Let $f : X' - - \rightarrow X$ be a birational map of compact complex manifolds with nef canonical bundles K_X and $K_{X'}$, respectively. If $\alpha \in H^2(X, \mathbb{R})$ is a class such that $\int_C \alpha > 0$ and $\int_{C'} f^*(\alpha) > 0$ for all rational curves $C \subset X$ and $C' \subset X'$, then f extends to an isomorphism $X \cong X'$.*

Proof. Let $\pi : Z \rightarrow X$ be a sequence of blow-ups resolving f, i.e. there exists a commutative diagram of the form

$$\begin{array}{ccc} & Z & \\ \pi' \swarrow & & \searrow \pi \\ X' & - \overset{f}{-} \rightarrow & X \end{array}$$

The assumption that K_X and $K_{X'}$ are nef implies that f induces an isomorphism on the complement of certain codimension ≥ 2 analytic subsets in X and X' and that any exceptional divisor E_i of $\pi : Z \to X$ is also exceptional for $\pi' : Z \to X'$. Recall, that there is a positive combination $\sum n_i E_i$, i.e. $n_i \in \mathbb{N}_{>0}$, such that $-\sum n_i E_i$ is π-ample.

Using the two direct sum decompositions $H^2(Z,\mathbb{R}) = \pi^* H^2(X,\mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$ and $H^2(Z,\mathbb{R}) = \pi'^* H^2(X',\mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$, any class $\beta \in H^2(X,\mathbb{R})$ can be written as $\pi^*\beta = \pi'^*\beta' + \sum a_i[E_i]$, where $\beta' = f^*\beta$ and $a_i \in \mathbb{R}$.

Now let $\alpha' := f^*(\alpha)$, where α is as in the proposition. In a first step we show that all coefficients a_i in $\pi^*\alpha = \pi'^*\alpha' + \sum a_i[E_i]$ are non-negative.

Assume that this is not the case, i.e. $a_1, \ldots, a_k < 0$, $a_{k+1}, \ldots, a_\ell \geq 0$ for some $k \geq 1$. We can assume that $-(a_1/n_1) = \max_{i=1,\ldots k}\{-(a_i/n_i)\}$. Let $C' \subset E_1$ be a general rational curve contracted under $\pi : E_1 \to X$, which exists as the general fibre of $\pi : E_1 \to X$ is unirational. As C' is general, one has $E_i.C' \geq 0$ for all $i > 1$ and hence

$$\begin{array}{rclcl} -\sum_{i=1}^{\ell} a_i(E_i.C') & \leq & -\sum_{i=1}^{k} a_i(E_i.C') & = & -\frac{a_1}{n_1} n_1(E_1.C') + \sum_{i=2}^{k}(-\frac{a_i}{n_i}) n_i(E_i.C') \\ & \leq & (-\frac{a_1}{n_1})\sum_{i=1}^{k} n_i(E_i.C') & \leq & (-\frac{a_1}{n_1})\sum_{i=1}^{\ell} n_i(E_i.C') \\ & < & 0. & & \end{array}$$

On the other hand, $0 = \pi^*\alpha.C' = (\pi'^*\alpha' + \sum_{i=1}^{\ell} a_i[E_i]).C' \geq \sum_{i=1}^{\ell} a_i(E_i.C')$. Contradiction.

Interchanging the rôle of α and α', one proves analogously $a_i \leq 0$ for all i. Hence, $a_i = 0$ for all i, i.e. $\pi'^*\alpha' = \pi^*\alpha$.

In a second step, we show that for all exceptional divisors the two contractions $\pi : E \to X$ and $\pi' : E \to X'$ coincide. Assume that there exists a rational curve $C' \subset E_i$ contracted by π, such that $\pi' : C' \to X'$ is finite. Since $\pi'^*\alpha = \pi^*\alpha$, this yields the contradiction $0 < \pi'^*\alpha'.C' = \pi^*\alpha.C' = 0$. Analogously, one excludes the case that there exists a rational curve C contracted by π' with $\pi|_C$ finite. As the fibres of $\pi'|_{E_i}$ and $\pi|_{E_i}$ are covered by rational curves, this shows that the two projections do coincide.

But if all contractions $\pi|_{E_i}$ and $\pi'|_{E_i}$ coincide, the birational map $X' - - \to X$ extends to an isomorphism. □

Note that the proposition in particular applies to the case where X and X' admit holomorphic symplectic structures (e.g. X and X' are compact hyperkähler manifolds), as in this case K_X and $K_{X'}$ are even trivial. The same arguments also prove:

Corollary 2.2 — *Let $f : X' - - \to X$ be a birational map of compact complex manifolds with nef canonical bundles. If $\alpha \in H^2(X,\mathbb{R})$ is a class that is positive on all (rational) curves $C \subset X$, then $\int_{C'} f^*(\alpha) > 0$ for all irreducible (rational) curves $C' \subset X'$ that are not contained in the exceptional locus of f.*

Proof. Under the given assumptions the arguments above show that the coefficients a_i in $\pi'^*\alpha' = \pi^*\alpha + \sum a_i[E_i]$ are non-negative. If $C' \subset X'$ is an irreducible (rational) curve

not contained in the exceptional locus of f, then its strict transform $\bar{C}' \subset Z$ exists and $\int_{C'} \alpha' = \int_{\bar{C}'} \pi'^* \alpha' = \int_{\bar{C}'} (\pi^* \alpha + \sum a_i E_i) > \int_{\bar{C}'} (\sum a_i E_i) \geq 0$. □

For completeness sake we recall the following results [11, Prop. 5.1, Cor. 5.2]:

Proposition 2.3 — *Let X be a compact hyperkähler manifold and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a very general element of the positive cone $\mathcal{C}_X$. Then there exist two smooth proper families $\mathcal{X} \to S$ et $\mathcal{X}' \to S$ of compact hyperkähler manifolds over an one-dimensional disk S and a birational map $\tilde{f} : \mathcal{X}' - - \to \mathcal{X}$ compatible with the projections to S, such that $\tilde{f}$ induces an isomorphism $\mathcal{X}'|_{S \setminus \{0\}} \cong \mathcal{X}|_{S \setminus \{0\}}$, the special fibre $\mathcal{X}_0$ is isomorphic to X, and $\tilde{f}^* \alpha$ is a Kähler class on $\mathcal{X}'_0$.* □

As S is contractible, $H^2(X, \mathbb{R}) \cong H^2(\mathcal{X}, \mathbb{R})$ and $\tilde{f}^* \alpha \in H^2(\mathcal{X}', \mathbb{R}) = H^2(\mathcal{X}'_0, \mathbb{R})$. If $\mathcal{X} \leftarrow \mathcal{Z} \to \mathcal{X}'$ resolves the birational map $\tilde{f}$, then $\tilde{f}^* \alpha = [\Gamma]_*(\alpha)$, where $\Gamma = \mathrm{im}(\mathcal{Z}_0 \to \mathcal{X}_0 \times \mathcal{X}'_0)$.

Corollary 2.4 — *Let X be a compact hyperkähler manifold and let $\alpha \in \mathcal{C}_X$ be a very general class. Then there exists a compact hyperkähler manifold X' and an effective cycle $\Gamma = Z + \sum Y_i \subset X \times X'$, such that $X \leftarrow Z \to X'$ defines a birational map $X' - - \to X$, $[\Gamma]_* : H^*(X) \cong H^*(X')$ is a ring isomorphism compatible with q_X and $q_{X'}$, and $[\Gamma]_*(\alpha)$ is a Kähler class on X'.* □

By definition, the very general classes in the positive cone $\mathcal{C}_X$ are cut out by a countable number of nowhere dense closed subsets. In fact, it suffices to assume that $q_X(\alpha, \beta) \neq 0$ for all integral class $0 \neq \beta \in H^2(X, \mathbb{Z})$. Note that there is a slight abuse of notation here. The component Z in Γ really occurs with multiplicity one, whereas any other component Y_i may occur several times in $\sum Y_i$. Of course, the components Y_i correspond to the exceptional divisors of the birational correspondence $\mathcal{X} \leftarrow \mathcal{Z} \to \mathcal{X}'$.

The main result of this section is the following

Theorem 2.5 — *Let X and X' be compact hyperkähler manifolds and let $f : X' - - \to X$ be a birational map. Then there exist smooth proper families $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ over a one-dimensional disk S with the following properties i) The special fibres are $\mathcal{X}_0 \cong X$ and $\mathcal{X}'_0 \cong X'$. ii) There exists a birational map $\tilde{f} : \mathcal{X}' - - \to \mathcal{X}$ which is an isomorphism over $S \setminus \{0\}$, i.e. $\tilde{f} : \mathcal{X}'|_{S \setminus \{0\}} \cong \mathcal{X}|_{S \setminus \{0\}}$, and which coincides with f on the special fibre, i.e. $\tilde{f}_0 = f$.*

This generalizes a result of [11], where X and X' were assumed to be projective. In an earlier version [10] of this result we had moreover assumed that X and X' are isomorphic in codimension two. The projective arguments in the previous proofs, e.g. comparing sections of line bundles on X and X', are here replaced by arguments from twistor theory.

Proof. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a class associated with a very general $\alpha' \in \mathcal{C}_{X'}$ such that $C.\alpha' > 0$ for all (rational) curves $C' \subset X'$, e.g. a very general $\alpha' \in \mathcal{K}_{X'}$. Thus, α is a very

general class in $\mathcal{C}_X$ and, hence, there exist two morphisms $\mathcal{X} \to S$, $\mathcal{X}' \to S$ as in Prop. 2.3. It suffices to show that $\mathcal{X}'_0 \cong X'$ and that under this isomorphism $\tilde{f}_0$ and f coincide.

The cycle $\Gamma := \mathrm{im}(\mathcal{Z}_0 \to X \times \mathcal{X}'_0)$ decomposes into $\Gamma = Z + \sum Y_i$, where the $Y_i \subset X \times \mathcal{X}'_0$ correspond to the exceptional divisors D_i of $\mathcal{X} \stackrel{\pi}{\leftarrow} \mathcal{Z} \stackrel{\pi'}{\to} \mathcal{X}'$ and $X \leftarrow Z \to \mathcal{X}'_0$ is a birational correspondence. If the codimension of the image of D_i under $\pi' : D_i \to \mathcal{X}'_0$ is at least two in $\mathcal{X}'_0$, then $[Y_i]_* : H^2(X) \to H^2(\mathcal{X}'_0)$ is trivial. If this is the case for all i, then $\beta := [\Gamma]_*(\alpha) = [Z]_*(\alpha)$. Hence, in this case, under the birational correspondence $\mathcal{X}'_0 \leftarrow Z \to X - - \to X'$ the class α' on X' is mapped to the Kähler class β on $\mathcal{X}'_0$. Prop. 2.1 then shows that the birational map $\mathcal{X}'_0 - - \to X'$ can be extended to an isomorphism. Clearly, $\tilde{f}_0$ corresponds to f.

Thus, it suffices to show that for all exceptional divisors D_i the image $\pi'(D_i) \subset \mathcal{X}'_0$ has codimension at least two.

Let $D_1, \ldots, D_k$ be those exceptional divisors for which $\pi'(D_i) \subset \mathcal{X}'_0$ is of codimension one. We will first show that also $\pi(D_1), \ldots, \pi(D_k) \subset X$ are of codimension one. In fact, $\pi(D_i) \subset \mathcal{X}_0$ is a divisor if and only if $\pi'(D_i) \subset \mathcal{X}'_0$ is one. In order to prove this, we use that up to a non-trivial scalar $\pi^*\sigma|_{D_i} = \pi'^*\sigma'|_{D_i}$, where σ and σ' are non-trivial two-forms on X and $\mathcal{X}_0$. Hence, for any point $z \in D_i$ the homomorphisms $\pi^*\sigma : \mathcal{T}_{D_i}(z) \to \mathcal{T}_X(\pi(z)) \cong \Omega_X(\pi(z)) \to \Omega_{D_i}(z)$ and $\pi'^*\sigma' : \mathcal{T}_{D_i}(z) \to \mathcal{T}_{\mathcal{X}'_0}(\pi'(z)) \cong \Omega_{\mathcal{X}'_0}(\pi'(z)) \to \Omega_{D_i}(z)$ coincide. If, for instance, $\mathrm{codim}(\pi(D_i)) = 1$, then $\dim(\ker(\mathcal{T}_{D_i}(z) \to \mathcal{T}_X(z))) = 1$ for $z \in D_i$ general and, therefore, $\dim(\ker \pi^*\sigma) \leq 2$. Hence, $\dim(\ker \pi'^*\sigma') \leq 2$. This yields that $\mathrm{codim}(\pi'(D_i)) = 1$ or $\ker(\mathcal{T}_{D_i}(z) \to \mathcal{T}_X(\pi(z))) \subset \ker(\mathcal{T}_{D_i}(z) \to \mathcal{T}_{\mathcal{X}'_0}(\pi'(z)))$, but the latter is excluded, for the general fibre of $\pi : D_i \to X$ is not contracted by π'. (In fact, this is only true for those divisors D_i on which π and π' differ, but if they do not differ, then of course $\mathrm{codim}(\pi(D_i)) = \mathrm{codim}(\pi'(D_i))$.)

Next, choose general irreducible (rational) curves $C_i \subset D_i$ $(i = 1, \ldots, k)$ contracted under $\pi' : D_i \to \mathcal{X}'_0$. As the exceptional locus $Sing(f^{-1})$ of the birational map $f^{-1} : X - - \to X'$ is of codimension at least two and $\pi(D_i) \subset X$ is a divisor, the image $\pi(C_i)$ will not be contained in $Sing(f^{-1})$. Moreover, we can assume that the C_i's $(i = 1, \ldots, k)$ do not meet D_j for $j > k$. This follows from $\mathrm{codim}(\pi'(D_j)) \geq 2$ for $j > k$. Now, use $H^2(\mathcal{Z}, \mathbb{R}) = \pi^* H^2(\mathcal{X}, \mathbb{R}) \oplus \bigoplus \mathbb{R}[D_i] = \pi^*\Gamma(S, \underline{H}^2(X, \mathbb{R})) \oplus \bigoplus \mathbb{R}[D_i]$ and $H^2(\mathcal{Z}, \mathbb{R}) = \pi'^* H^2(\mathcal{X}', \mathbb{R}) \oplus \bigoplus \mathbb{R}[D_i] = \pi^*\Gamma(S, \underline{H}^2(\mathcal{X}'_0, \mathbb{R})) \oplus \bigoplus \mathbb{R}[D_i]$ and write $\pi^*\tilde{\alpha} = \pi'^*\tilde{\beta} + \sum_{i=1}^{\ell} a_i[D_i]$, where $\tilde{\alpha} \in \Gamma(S, \underline{H}^2(X, \mathbb{R}))$ and $\tilde{\beta} \in \Gamma(S, \underline{H}^2(\mathcal{X}'_0, \mathbb{R}))$ are the constant sections α and $\beta := [\Gamma]_*(\alpha)$. As in the proof of Prop. 2.1 one shows that β Kähler implies $a_i \geq 0$. Also note that there exists a positive linear combination $\sum_{i=1}^{\ell} m_i D_i$ which is π'-negative. We may assume that $(a_1/m_1) = \max_{i=1,\ldots,k}\{(a_i/m_i)\}$. Using Cor. 2.2 all this yields the contradiction

$$
\begin{array}{rclcl}
0 \leq \pi^*\alpha.C_1 & = & (\pi'^*\beta + \sum_{i=1}^{\ell} a_i D_i).C_1 & = & \sum_{i=1}^{\ell} a_i(D_i.C_1) \\
& = & \sum_{i=1}^{k} a_i(D_i.C_1) & \leq & \frac{a_1}{m_1} m_1(D_1.C_1) + \frac{a_1}{m_1} \sum_{i=2}^{k} m_i(D_i.C_1) \\
& = & \frac{a_1}{m_1} \sum_{i=1}^{k} m_i(D_i.C_1) & = & \frac{a_1}{m_1}(\sum_{i=1}^{\ell} m_i D_i).C_1 \\
& < & 0 & &
\end{array}
$$

Hence, the varieties $\pi'(D_i) \subset \mathcal{X}'_0$ are all of codimension at least two. $\square$

Note that there are two kinds of rational curves in the proof. Those, that sweep out a divisor in $\mathcal{X}'_0$ and those that a priori do not. The former correspond to the general fibres of the projection $D_i \to X$ of an exceptional divisor, such that $\pi'(D_i) \subset \mathcal{X}'_0$ is a divisor, and the latter ones to those that are created by the birational correspondence $\mathcal{X}_0 - - \to X'$ itself. In Sect. 3 we will see that if α is positive on all rational curves, then it is automatically a Kähler class. The same idea proves that the positivity of α on all rational curves that sweep out a divisor is sufficient to conclude that α is in the "birational Kähler cone". In Sect. 4.2 this is phrased by assuming α to be positive on all uniruled divisors.

As mentioned in the introduction the theorem immediately yields

Corollary 2.6 — *Let X and X' be birational compact hyperkähler manifolds. Then the Hodge structures of X and X' are isomorphic. In particular, Hodge and Betti numbers of X and X' coincide. Moreover, $H^*(X,\mathbb{Z}) \cong H^*(X',\mathbb{Z})$ as graded rings.* $\square$

Corollary 2.7 — *Let X and X' be birational compact hyperkähler manifolds of dimension $2n$. Then there exists a cycle $\Gamma = Z + \sum Y_i \subset X \times X'$ of dimension $2n$, such that*
i) The correspondence $X' \leftarrow Z \to X$ is the given birational map $X' - - \to X$.
ii) The correspondence $[\Gamma]_* : H^*(X,\mathbb{Z}) \cong H^*(X',\mathbb{Z})$ is a ring isomorphism and $[\Gamma]_*$ and $[Z]_*$ coincide on $H^2(X,\mathbb{Z})$.
iii) The isomorphism $[\Gamma]_*$ is compatible with q, i.e. $q_{X'}([\Gamma]_*(\ )) = q_X(\ )$.
iv) The isomorphism $[\Gamma]_*$ is an involution, i.e. $[\Gamma]_*[\Gamma]_* = \mathrm{id}$. $\square$

3 The Kähler cone

Let us recall the precise definition of the Kähler cone of a compact Kähler manifold.

Definition 3.1 — *For a complex manifold X the Kähler cone $\mathcal{K}_X$ is the set of all classes $\alpha \in H^{1,1}(X,\mathbb{R})$ that can be represented by a closed positive $(1,1)$-form.*

Obviously, any ample line bundle L on X defines a class $c_1(L)$ in $\mathcal{K}_X$. Thus, the ample cone, i.e. the cone spanned by all classes $c_1(L)$ with L ample, is contained in $\mathcal{K}_X$. On the other hand, if X is projective of dimension m and the canonical bundle K_X is trivial, then a line bundle on X is ample if and only if $\int_X c_1^m(L) > 0$ and $\int_C c_1(L) > 0$ for any curve $C \subset X$ (cf. [11, Prop. 6.3]). In other words, the ample cone can be completely described by these two conditions. For the case of a compact hyperkähler manifold (not necessarily projective) there is a similar description for the Kähler cone $\mathcal{K}_X$.

Proposition 3.2 — *Let X be a compact hyperkähler manifold. A class $\alpha \in H^{1,1}(X,\mathbb{R})$ is contained in the closure $\overline{\mathcal{K}}_X$ of the Kähler cone $\mathcal{K}_X \subset H^{1,1}(X,\mathbb{R})$ if and only if $\alpha \in \overline{\mathcal{C}}_X$ and $\int_C \alpha \geq 0$ for any rational curve $C \subset X$.*

Proof. Let $\alpha \in \overline{\mathcal{K}}_X$. Then, obviously, $\int_C \alpha \geq 0$ for any curve $C \subset X$ and, since $\mathcal{K}_X \subset \mathcal{C}_X$, also $\alpha \in \overline{\mathcal{C}}_X$.

Now let $\alpha \in \overline{\mathcal{C}}_X$ and $\int_C \alpha \geq 0$ for any rational curve $C \subset X$. If ω is any Kähler class, then $\alpha + \varepsilon\omega \in \mathcal{C}_X$ and $\int_C(\alpha + \varepsilon\omega) > 0$. For ω and ε general, $\alpha + \varepsilon\omega$ is general as well and if $\alpha + \varepsilon\omega \in \mathcal{K}_X$ for ω general and ε small, then $\alpha \in \overline{\mathcal{K}}_X$. Thus, it suffices to show that any general class $\alpha \in \mathcal{C}_X$ with $\int_C \alpha > 0$ for any rational curve $C \subset X$ is actually Kähler, i.e. contained in $\mathcal{K}_X$.

The proof of 2.5 shows that for such a class α there exists a birational map $X - - \to \mathcal{X}'_0$, such that α corresponds to a Kähler class on $\mathcal{X}'_0$. By Prop. 2.1 this readily implies $X \cong \mathcal{X}'_0$ and, hence $\alpha \in \mathcal{K}_X$. (The situation corresponds to the trivial case $X = X'$ in the proof of 2.5.). □

A weaker form of the following result was already proved in [11, Cor. 5.6].

Corollary 3.3 — *Let X be a compact hyperkähler manifold not containing any rational curve. Then $\mathcal{C}_X = \mathcal{K}_X$.* □

The proposition can be made more precise: If $\alpha \in \partial\overline{\mathcal{K}}_X$, then $\alpha \in \partial\overline{\mathcal{C}}_X$, i.e. $q_X(\alpha) = 0$, or there exists a class $0 \neq \beta \in H^{1,1}(X, \mathbb{Z})$ othogonal to α. This follows from the remark after Cor. 2.4. If neither $q_X(\alpha) = 0$ nor $q_X(\alpha, \beta) = 0$ for some $0 \neq \beta \in H^{1,1}(X, \mathbb{Z})$, then α is already general and the above proof applies directly to α (without adding $\varepsilon\omega$) and shows $\alpha \in \mathcal{K}_X$. We therefore expect an affirmative answer to the following

Question — *Is the Kähler cone $\mathcal{K}_X$ of a compact hyperkähler manifold X the set of classes $\alpha \in \mathcal{C}_X$ with $\int_C \alpha > 0$ for all rational curves $C \subset X$?*

In dimension two this can be proved:

Corollary 3.4 — *Let X be a K3 surface. Then the Kähler cone $\mathcal{K}_X$ is the set of all classes $\alpha \in \mathcal{C}_X$ such that $\int_C \alpha > 0$ for all smooth rational curves $C \subset X$.*

Proof. Let C be an irreducible rational curve. Then $C^2 \geq 0$ or C is smooth and $C^2 = -2$. Let $\alpha \in \mathcal{C}_X$. If $C^2 \geq 0$, then $\alpha.C \geq 0$. Hence, the proposition shows $\overline{\mathcal{K}}_X = \{\alpha \in \mathcal{C}_X | \alpha.C \geq 0 \text{ for every smooth rational curve}\}$. It suffices to show that for every class $\alpha \in \partial\overline{\mathcal{K}}_X \cap \mathcal{C}_X$ there exists a smooth rational curve C with $\alpha.C = 0$. Let $\{C_i\}$ be a series of smooth irreducible rational curves, such that $\alpha.C_i \to 0$. Let $\alpha_1 = \alpha, \alpha_2 \ldots, \alpha_{20}$ be an orthogonal base of $H^{1,1}(X, \mathbb{R})$. We can even require $\alpha_{i>1}^2 = -1$ and $\alpha_i.\alpha_{j \neq i} = 0$. For the coefficients of $[C_i] = \lambda_i\alpha + \sum \mu_{ij}\alpha_j$ one then concludes $\lambda_i \to 0$ and $\sum \mu_{ij}^2 \to 2$. Thus, the set of classes $[C_i]$ is contained in a compact ball and, hence, there is only a finite number of them. The assertion follows directly form this. □

Of course, the result is well-known, but the original arguments are based on the Global Torelli Theorem, which seems, at least at the moment, out of reach in higher dimensions.

The above proof shows that it would suffice to find a lower bound for $q_X([C_i])$ for all rational curves C_i such that $\int_{C_i} \alpha \to 0$. This should also shed some light on the question whether $\partial\mathcal{K}_X$ is, away from $\partial\mathcal{C}_X$, locally a finite polyhedron.

Note that also for the two other classes of compact Kähler manifolds with trivial canonical bundle, compact complex tori and Calabi-Yau manifolds, there exist descriptions of the Kähler cone similar to the one for hyperkähler manifolds presented above. E.g. for a Calabi-Yau manifold X the closure of the Kähler cone $\overline{\mathcal{K}}_X$ is the set of all classes $\alpha \in H^2(X, \mathbb{R})$ such that $\int_X \alpha^m \geq 0$ and $\int_C \alpha \geq 0$ for all (not necessarily rational) curves $C \subset X$. This is due to the fact that on a Calabi-Yau manifold the ample classes span the Kähler cone.

4 The birational Kähler cone

Let X be a compact hyperkähler manifold and let $f : X - - \to X'$ be a birational map to another compact hyperkähler manifold X'. Via the natural isomorphism $H^{1,1}(X', \mathbb{R}) \cong H^{1,1}(X, \mathbb{R})$ the Kähler cone $\mathcal{K}_{X'}$ of X' can also be considered as an open subset of $H^{1,1}(X, \mathbb{R})$ and, due to [11, Lemma 2.6], in fact as an open subset of $\mathcal{C}_X$. The union of all those open subsets is called the birational Kähler cone of X.

Definition 4.1 — *The birational Kähler cone $\mathcal{BK}_X$ is the union $\bigcup_{f:X--\to X'} f^*(\mathcal{K}_{X'})$, where $f : X - - \to X'$ runs through all birational maps $X - - \to X'$ from X to another compact hyperkähler manifold X'.*

Note that $\mathcal{BK}_X$ is in fact a disjoint union of the $f^*(\mathcal{K}_{X'})$, i.e. if $f_1^*(\mathcal{K}_{X_1})$ and $f_2^*(\mathcal{K}_{X_2})$ have a non-empty intersection then they are equal, and that $\mathcal{BK}_X$ is in general not a cone in $\mathcal{C}_X$. As it will turn out, its closure $\overline{\mathcal{BK}}_X$ is a cone. Notice that this is far from being obvious. E.g. if X and X' are birational compact hyperkähler manifolds and α and α' are Kähler classes on X and X', respectively, there is a priori no reason why every generic class α'' contained in the segment joining α and α' should be a Kähler class on yet another birational compact hyperkähler manifold X''. In this point the theory resembles the theory of Calabi-Yau threefolds [12]. The aim of this section is to provide two descriptions of $\mathcal{BK}_X$ (or rather its closure) similar to the two descriptions of $\mathcal{K}_X$ given by Def. 3.1 and Prop. 3.2.

We begin with a geometric description of the birational Kähler cone analogous to 3.2. Here the rational curves are replaced by uniruled divisors.

Proposition 4.2 — *Let X be a compact hyperkähler manifold. Then $\alpha \in H^{1,1}(X, \mathbb{R})$ is in the closure $\overline{\mathcal{BK}}_X$ of the birational Kähler cone $\mathcal{BK}_X$ if and only if $\alpha \in \overline{\mathcal{C}}_X$ and $q_X(\alpha, [D]) \geq 0$ for all uniruled divisors $D \subset X$.*

Proof. Let $\alpha \in \mathcal{BK}_X$ and $f : X' - - \to X$ be a birational map such that $f^*(\alpha) \in \mathcal{K}_{X'}$. By [11, Lemma 2.6] for any divisor $D \subset X$ we have $q_X(\alpha, [D]) = q_{X'}(f^*\alpha, f^*[D])$. Since $f^*\alpha \in \mathcal{K}_{X'}$ and $f^*[D] = [f^*D]$ is effective, $q_X(\alpha, [D]) > 0$. Hence, any $\alpha \in \overline{\mathcal{BK}}_X$ is contained

in $\overline{\mathcal{C}}_X$ and is non-negative on any effective divisor $D \subset X$ and, in particular, on uniruled ones.

For the other direction, let $\alpha \in \mathcal{C}_X$ be a general class with $q_X(\alpha, [D]) > 0$ for all uniruled divisors $D \subset X$. We will show that $\alpha \in \mathcal{BK}_X$. This would be enough to prove the assertion.

Using Prop. 2.3 we find an effective cycle $\Gamma = Z + \sum Y_i \subset X \times X'$, where X' is another compact hyperkähler manifold, Z defines a birational map $f : X' - - \to X$, and $[\Gamma]_*(\alpha) \in \mathcal{K}_{X'}$. It suffices to show that $[\Gamma]_*(\alpha) = [Z]_*(\alpha) = f^*\alpha$. Clearly, if $\pi(Y_i) \subset X$ is of codimension at least two, then $[Y_i]_*(\alpha) = 0$. Here, $\pi : Y_i \to X$ and $\pi' : Y_i \to X'$ denote the first and second projection, respectively.

Assume that $Y_1, \ldots, Y_k$ are those components with $\pi(Y_i) \subset X$ of codimension one (or, equivalently $\pi'(Y_i) \subset X'$ of codimension one, as was explained in the proof of 2.5) and $k > 0$. Firstly, we claim that $\sum_{i=1}^k [Y_i]_*(\sum_{i=1}^k [\pi(Y_i)]) = -\sum_{i=1}^k b_i'[\pi'(Y_i)]$ with $b_i' > 0$. In order to see this, recall that $\Gamma = Z + \sum Y_i$ is the special fibre $\mathcal{Z}_0$ of $\mathcal{Z} \to S$. Hence, $\mathcal{O}_{\mathcal{Z}}(\Gamma)|_\Gamma$ is trivial and, therefore, $\mathcal{O}_{\mathcal{Z}}(-\sum Y_i)|_{C_j} \cong \mathcal{O}_{\mathcal{Z}}(Z)|_{C_j}$, where C_j is a generic fibre of $\pi' : Y_j \to X'$. As Y_j meets, but is not contained in Z, this yields $\deg(\mathcal{O}_{\mathcal{Z}}(Z)|_{C_j}) > 0$ and hence $-\sum_{i=1}^k (Y_i.C_j) > 0$. But now $\sum_{j=1}^k [Y_j]_*(\sum_{i=1}^k [\pi(Y_i)]) = \sum_{j=1}^k (\sum_{i=1}^k (Y_i.C_j))[\pi'(Y_j)]$ and we define $b_j' := -\sum_{i=1}^k (Y_i.C_j) > 0$. Analogously, one has $\sum_{i=1}^k [Y_i]_*(\sum_{i=1}^k [\pi'(Y_i)]) = -\sum_{i=1}^k b_i[\pi(Y_i)]$ with $b_i > 0$. Then, the assumption on α and the property $[\Gamma]_*(\alpha) \in \mathcal{K}_{X'}$ yield the following contradiction

$$
\begin{aligned}
& q_X(\alpha, \textstyle\sum_{i=1}^k [\pi(Y_i)]) = q_{X'}([\Gamma]_*(\alpha), [\Gamma]_*(\sum_{i=1}^k [\pi(Y_i)])) \\
& = q_{X'}([\Gamma]_*(\alpha), [Z]_*(\textstyle\sum_{i=1}^k [\pi(Y_i)]) + (\sum_{i=1}^k [Y_i]_*)(\sum_{i=1}^k [\pi(Y_i)]) \\
& = q_{X'}([\Gamma]_*(\alpha), \textstyle\sum_{i=1}^k [\pi'(Y_i)]) - q_{X'}([\Gamma]_*(\alpha), \sum_{i=1}^k b_i'[\pi'(Y_i)]) \\
& < q_{X'}([\Gamma]_*(\alpha), \textstyle\sum_{i=1}^k [\pi'(Y_i)]) \\
& = q_X([\Gamma]_*[\Gamma]_*(\alpha), [\Gamma]_*(\textstyle\sum_{i=1}^k [\pi'(Y_i)])) \\
& = q_X(\alpha, \textstyle\sum_{i=1}^k [\pi(Y_i)]) + q_X(\alpha, (\sum_{i=1}^k [Y_i]_*)(\sum_{i=1}^k [\pi'(Y_i)])) \\
& < q_X(\alpha, \textstyle\sum_{i=1}^k [\pi(Y_i)]).
\end{aligned}
$$

Here we use that $\pi(Y_i)$ is uniruled, as it is a component of the exceptional locus of the birational map $\mathcal{X}' \leftarrow \mathcal{Z} \to \mathcal{X}$. Indeed, any exceptional divisor D_i of $\pi : \mathcal{Z} \to \mathcal{X}$ is also exceptional for $\pi' : \mathcal{Z} \to \mathcal{X}'$. If the two projections on D_i differ then the images under π and π' are uniruled. If the two projections on D_i are generically equal, than the image $\pi(D_i)$ is contained in the uniruled image of an exceptional divisor for which this is not the case. But in our case the dimension of $\pi(Y_i)$, $i = 1, \ldots, k$ is already maximal. □

Corollary 4.3 — *If X is a compact hyperkähler manifold not containing any uniruled divisor, then $\mathcal{BK}_X$ is dense in $\mathcal{C}_X$.* □

In order to obtain a description of the birational Kähler cone analogous to the definition(!) of $\mathcal{K}_X$ (cf. 3.1), we need the following

Lemma 4.4 — *Let X be a compact Kähler manifold of dimension N. A class $\alpha \in H^{1,1}(X,\mathbb{R})$ can be represented by a closed positive $(1,1)$-current if and only if $\int \alpha\gamma \geq 0$ for all classes $\gamma \in H^{N-1,N-1}(X,\mathbb{R})$ that can be represented by a closed $2N-2$-form whose $(N-1,N-1)$-part is positive.*

Proof. A $(1,1)$-current is a continous linear function $T : \mathcal{A}^{N-1,N-1}(X)_{\mathbb{R}} \to \mathbb{R}$. It is called positive if $T(\gamma) \geq 0$ for any positive form $\gamma \in \mathcal{A}^{N-1,N-1}(X)_{\mathbb{R}}$. Recall that a $(N-1,N-1)$-form is positive if and only if it locally is a positive linear combination of forms of the type $i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_{N-1} \wedge \overline{\alpha}_{N-1}$, where the α_i's are $(1,0)$-forms. Note that in Def. 3.1 we used 'positive' in the sense of 'strictly positive'. A $(1,1)$-current $T : \mathcal{A}^{N-1,N-1}(X)_{\mathbb{R}} \to \mathbb{R}$ is closed if its extension by zero to a linear map $\mathcal{A}^{2N-2}(X)_{\mathbb{R}} \to \mathbb{R}$ is trivial on all exact forms. Note that a priori it does not suffice to show that the $(1,1)$-current is trivial on all exact $(N-1,N-1)$-forms.

The idea is to construct first a continous linear function T_0 on all $(N-1,N-1)$-forms $\gamma^{N-1,N-1}$, where $\gamma^{N-1,N-1}$ is the $(N-1,N-1)$-part of a closed $(2N-2)$-form γ. This T_0 shall have the properties: *i)* $T_0(\gamma^{N-1,N-1}) = 0$ for any exact $(2N-2)$-form γ and *ii)* $T_0(\gamma^{N-1,N-1}) \geq 0$ for any $\gamma^{N-1,N-1}$ that is in addition positive. This T_0 will then be extended to a closed positive $(1,1)$- current.

The extension argument uses the following general result [6, Ch. II]: Let A be a topological vector space, $B \subset A$ a subspace and $C \subset A$ a convexe cone, such that $C \cap (-C) = 0$ and $B \cap \overset{\circ}{C} \neq \emptyset$. Then, any continous linear function $T_0 : B \to \mathbb{R}$ with $T_0|_{B\cap C} \geq 0$ can be extended to a continous linear function $T : A \to \mathbb{R}$ with $T|_C \geq 0$.

In our situation we let $A := \mathcal{A}^{N-1,N-1}(X)_{\mathbb{R}}/\{(d\eta)^{N-1,N-1} \mid \eta \in \mathcal{A}^{2N-3}(X)_{\mathbb{R}}\}$ and C be the image of the convexe cone of all positive forms in $\mathcal{A}^{N-1,N-1}(X)_{\mathbb{R}}$. Clearly, C is again a convex cone. In order to see that $C \cap (-C) = 0$, one argues that for $\gamma_1, \gamma_2 \in \mathcal{A}^{N-1,N-1}(X)_{\mathbb{R}}$ positive with $\overline{\gamma}_1 = -\overline{\gamma}_2$ in A, one has $0 \leq \int \gamma_1\omega = \int((d\eta)^{N-1,N-1} - \gamma_2)\omega = \int(d\eta - \gamma_2)\omega = -\int \gamma_2\omega \leq 0$ for some $\eta \in \mathcal{A}^{2N-3}(X)_{\mathbb{R}}$, where ω is a Kähler form. Thus, $\int \gamma_1\omega = \int \gamma_2\omega = 0$ and, since ω is an inner point of the cone of all positive forms in $\mathcal{A}^{1,1}(X)_{\mathbb{R}}$, this implies $\gamma_1 = \gamma_2 = 0$. Last but not least, if $B := \{\gamma^{N-1,N-1} \in A \mid d(\gamma) = 0\}$ then $B \cap C$ contains ω^{N-1}, which is an inner point.

Therefore, in order to construct the closed positive current representing a class $\alpha \in \mathcal{C}_X$ one has to define $T_0 : B \to \mathbb{R}$ such that T_0 is non-negative on $B \cap C$. The obvious choice is $\gamma^{N-1,N-1} \mapsto \int \gamma^{N-1,N-1}\alpha = \int \gamma\alpha$, which only depends on the cohomology class of α as long as γ is closed and α is a closed $(1,1)$-form representing its cohomology class. It is well defined on B, because $\int \alpha\gamma^{N-1,N-1} = 0$ for all exact γ. It remains to show that $\int \alpha\gamma \geq 0$ for any positive $\gamma^{N-1,N-1} \in \mathcal{A}^{N-1,N-1}(X)_{\mathbb{R}}$ with $d(\gamma) = 0$. But this is ensured by the assumption.□

In analogy to the definition of $\mathcal{K}_X$ we can now prove the

Proposition 4.5 — *Let X be a compact hyperkähler manifold. The closure $\overline{\mathcal{BK}_X}$ of the birational Kähler cone $\mathcal{BK}_X$ is the set of all $\alpha \in H^{1,1}(X,\mathbb{R})$, such that $\alpha(\sigma\bar{\sigma})^{n-1} \in H^{2n-1,2n-1}(X,\mathbb{R})$*

can be represented by a closed $(4n-2)$*-form whose* $(2n-1,2n-1)$*-part is positive.*

Proof. Assume that $\alpha \in \overline{\mathcal{BK}}_X$, but that $\alpha(\sigma\bar{\sigma})^{n-1} \in H^{2n-1,2n-1}(X,\mathbb{R})$ cannot be represented by a closed $(4n-2)$-form with positive $(2n-1,2n-1)$-part. Since the set of classes $\gamma \in H^{2n-1,2n-1}(X,\mathbb{R})$ that can be represented by such forms is convex, there exists a class $\beta \in H^{1,1}(X,\mathbb{R})$, which is positive on those but negative on $\alpha(\sigma\bar{\sigma})^{n-1}$. Due to the above lemma β can be represented by a closed positive $(1,1)$-current.

Let $f : X' - - \rightarrow X$ be a birational map from another compact hyperkähler manifold X'. Then also $f^*\beta$ can be represented by a closed positive $(1,1)$-current (cf. [8]) and, therefore, $f^*\beta$ is positive on $\mathcal{K}_{X'}$ with respect to $q_{X'}$. Hence, β must be non-negative on $\overline{\mathcal{BK}}_X$, which contradicts $q_X(\alpha,\beta) = \int \alpha\beta(\sigma\bar{\sigma})^{n-1} < 0$.

On the other hand, if $\alpha(\sigma\bar{\sigma})^{n-1}$ can be represented by a closed $(4n-2)$-form with positive $(2n-1,2n-1)$-part, then $\int_D \alpha(\sigma\bar{\sigma})^{n-1} \geq 0$ for all divisors $D \subset X$. Prop. 4.2 then in particular yields $\alpha \in \overline{\mathcal{BK}}_X$. □

Remark 4.6 – It might be that in Lemma 4.4 it suffices to test the class α on those γ that can be represented by closed positive $(N-1,N-1)$-forms. The result in Prop. 4.5 would change accordingly.

Corollary 4.7 — *The closure of the birational Kähler cone* $\overline{\mathcal{BK}}_X$ *is dual to the cone of classes* $\alpha \in H^{1,1}(X,\mathbb{R})$ *that can be represented by closed positive* $(1,1)$*-currents.* □

5 Curves of negative square

For K3 surfaces it is well-known that an irreducible curve with negative self-intersection is a smooth rational curve. As the quadratic form q_X on an arbitrary higher-dimensional compact hyperkähler manifold is replacing the intersection pairing, one may wonder whether curves (or divisors) of negative square also have special geometric properties in higher dimensions.

First note that the quadratic form q_X on $H^{1,1}(X)$ for a compact hyperkähler manifold X also induces a quadratic form on $H^{2n-1,2n-1}(X)$, also denoted by q_X. Here, one can either use the natural duality isomorphism $H^{1,1}(X) \cong H^{2n-1,2n-1}(X)^\vee$ or the isomorphism $L : H^{1,1}(X) \cong H^{2n-1,2n-1}(X)$ given by taking the product with $(\sigma\bar{\sigma})^{n-1}$, where $0 \neq \sigma \in H^0(X,\Omega_X^2)$. Thus, we can speak of curves $C \subset X$ with square $q_X([C])$. For curves of negative square Prop. 3.2 yields:

Proposition 5.1 — *Let* X *be a compact hyperkähler manifold and* $C \subset X$ *be a curve with* $q_X([C]) < 0$*. Then there exists a class* $\alpha \in \mathcal{C}_X$ *and an irreducible rational curve* $C' \subset X$ *such that* $\alpha.C < 0$ *and* $\alpha.C' < 0$*.*

Proof. Let $\beta := L^{-1}([C])$. Then, $q_X(\beta) = q_X([C]) < 0$. If $\gamma \in \mathcal{K}_X$, then $q_X(\gamma, \beta) = \int_C \gamma > 0$. Therefore, the class $\alpha := (\gamma - (2q_X(\gamma,\beta)/q_X(\beta))\beta)$ satisfies $\int_C \alpha = q_X(\alpha, \beta) = -\gamma.C < 0$, $q_X(\alpha) = q_X(\gamma) > 0$, and $q_X(\alpha, \gamma) > 0$. Thus, the class α is contained in the positive cone $\mathcal{C}_X$, but $\alpha \notin \overline{\mathcal{K}}_X$. By 3.2 there exists an irreducible rational curve $C' \subset X$, such that $\alpha.C' < 0$. □

Corollary 5.2 — *In particular, if there are no irreducible rational curves of negative square in X, then all(!) curves have non-negative square.* □

Also, if $H^{2n-1,2n-1}(X, \mathbb{Q}) = \mathbb{Q}[C]$, where $C \subset X$ is of negative square, then there exists a rational curve $C' \subset X$ with $\mathbb{Q}[C'] = \mathbb{Q}[C]$. This (and the deformation theory of hyperkähler manifolds) suggests an affirmative answer to the following

Question — *Let $C \subset X$ be an irreducible curve in a compact hyperkähler manifold X with $q_X([C]) < 0$. Does there exist a rational curve $C' \subset X$ such that $\mathbb{Q}[C] = \mathbb{Q}[C']$ in $H^{2n-1,2n-1}(X, \mathbb{Q})$?*

Something slightly weaker can in fact be proved:

Proposition 5.3 — *Let C be an irreducible curve in a compact hyperkähler manifold X. If $q_X([C]) < 0$, then C is contained in a uniruled subvariety $Y \subset X$.*

Proof. As in the previous proof one finds a very general class $\alpha \in \mathcal{C}_X$ with $\alpha.C < 0$. Therefore, there exist two families $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ as in Prop. 2.3. Let $\mathcal{X} \leftarrow \mathcal{Z} \to \mathcal{X}'$ be a birational correspondence induced by the isomorphism $\mathcal{X}|_{S\setminus\{0\}} \cong \mathcal{X}'|_{S\setminus\{0\}}$. As the exceptional locus of the birational map $\mathcal{X} - - \to \mathcal{X}'$ is uniruled ($K_{\mathcal{X}}$ and $K_{\mathcal{X}'}$ are trivial), it suffices to show that $C \subset X = \mathcal{X}_0$ is in the exceptional locus of $\mathcal{X} \leftarrow \mathcal{Z} \to \mathcal{X}'$. If not, the strict transform $\bar{C} \subset \mathcal{Z}$ of C exists and $\tilde{\alpha}.C = \pi^*\tilde{\alpha}.\bar{C} = (\pi'^*\tilde{\alpha}' + \sum a_i[D_i]).\bar{C}$ (cf. Cor. 2.2). Here, we use the notation of the proof of Thm. 2.5, i.e. the D_i's are the exceptional divisors of $\mathcal{Z} \to \mathcal{X}$, $\tilde{\alpha}'_0 = \alpha'$ is a Kähler class on $\mathcal{X}'_0$, and the coefficients a_i's are positive. This yields the contradiction: $0 > \alpha.C = \pi'^*\tilde{\alpha}'.\tilde{C} + \sum a_i(D_i.C) > 0$. □

If one could replace uniruled by unirational in Prop. 5.3, we would obtain $\mathbb{Q}[C] = \mathbb{Q}[\sum a_i C_i]$, where the C_i's are rational curves, but the $a_i \in \mathbb{Z}$ might be negative (cf. [13]).

Analogously to Prop. 5.1 one can use the description of the birational Kähler cone (Prop. 4.2) to prove:

Proposition 5.4 — *Let X be a compact hyperkähler manifold and let $D \subset X$ be an effective divisor such that $q_X([D]) < 0$. Then there exists a class $\alpha \in \mathcal{C}_X$ and a uniruled divisor $D' \subset X$ such that $q_X(\alpha, [D]) < 0$ and $q_X(\alpha, [D']) < 0$. In particular, if X does not contain any uniruled divisor then there does not exist any divisor $D \subset X$ with $q_X([D]) < 0$.* □

References

[1] *Géométrie des surfaces K3: modules et périodes.* Séminaires Palaiseau. ed A. Beauville, J.-P. Bourguignon, M. Demazure. Astérisque 126 (1985).

[2] Batyrev, V. *On the Betti numbers of birationally isomorphic projective varieties with trivial canonical bundle.* London Math. Soc. Lect. Note Ser. 264 (1999).

[3] Beauville, A. *Variétés Kählériennes dont la première classe de Chern est nulle.* J. Diff. Geom. 18 (1983), 755-782.

[4] Bogomolov, F. A. *Hamiltonian Kähler manifolds.* Sov. Math. Dokl. 19 (1978), 1462-1465.

[5] Bogomolov, F. A. *On the cohomology ring of a simple hyperkähler manifold (on the results of Verbitsky).* GAFA 6(4) (1996), 612-618.

[6] Bourbaki, N. *Espace vectoriels topologiques.* Hermann, Paris (1964).

[7] Denef, J., Loeser, F. *Germs of arcs on singular algebraic varieties and motivic integration.* Invent. math. 135 (1999), 201-237.

[8] Fujiki, A. *A Theorem on Bimeromorphic Maps of Kähler Manifolds and Its Applications.* Publ. RIMS Kyoto Univ. 17 (1981), 735-754.

[9] Fujiki, A. *On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold.* Adv. Stud. Pure Math. 10 (1987), 105-165.

[10] Huybrechts, D. *Birational symplectic manifolds and their deformations.* J. Diff. Geom. 45 (1997), 488-513.

[11] Huybrechts, D. *Compact Hyperkähler Manifolds: Basic Results.* Invent. math. 135 (1999), 63-113. Erratum. math.AG/0106014

[12] Kawamata, Y. *Crepant blowing-up of 3-dimensional canonical singularities and its application to degeneration of surfaces.* Ann. Math. 127 (1988), 93-163.

[13] Kollár, J. *Rational curves on algebraic varieties.* Erg. Math. Grenzgebiete 3. Folge, 32 (1996).

[14] Morrison, D. *Beyond the Kähler cone.* Proc. Hirzebruch 65 Conference on Algebraic Geometry. Israel Math. Conf. Proc. 9 (1996), 361-376.

[15] Verbitsky, M. *Cohomology of compact hyperkähler manifolds and its applications.* GAFA 6(4) (1996), 602-611.

[16] Weil, A. *Introduction à l'étude des variétés kählériennes.* Hermann (1958).

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